

LAST SYZYGIES OF 1-GENERIC SPACES

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ABSTRACT. Consider a determinantal variety X of expected codimension defined by the maximal minors of a matrix M of linear forms. Eisenbud and Popescu have conjectured that 1-generic matrices M are characterised by the property that the syzygy ideals $I(s)$ of all last syzygies s of X coincide with I_X . In this note we prove a geometric version of this characterization, i.e. that M is 1-generic if and only if the syzygy varieties $\text{Syz}(s) = V(I(s))$ of all last syzygies have the same support as X .

1. INTRODUCTION

Syzygies and minimal free resolutions of projective varieties were introduced to algebraic geometry by Hilbert in 1890 to define what we now call the Hilbert polynomial. With Buchberger's algorithm [Buc70] and Schreyer's algorithm [Sch80] these minimal free resolutions and their syzygy spaces can be calculated explicitly in many examples. General results about the dimension of the syzygy spaces have been conjectured by Green [Gre84] and have resulted in the remarkable proof of Green's Conjecture for general canonical curves by Voisin [Voi02], [Voi03].

Much less is known about the geometric interpretation of individual syzygies. Two constructions of geometric objects associated to a p -th syzygy s are known, if s is a syzygy in the degree $d + 1$ linear strand of a projective variety $X \subset \mathbb{P}^n$ that does not lie on hypersurfaces of degree d :

- (1) the syzygy variety $\text{Syz}(s)$ which is the vanishing locus of a twisted $(p - 1)$ -form that can be associated to s .
- (2) the linear space $L(s)$ cut out by the linear forms involved in s . The codimension of $L(s)$ is also called the rank of s .

Both X and $L(s)$ are contained in $\text{Syz}(s)$. The syzygy varieties of low rank syzygies have been classified [Sch91], [Ehb94], [ES94], [vB01]. For example the syzygy variety of a rank $p + 1$ syzygy in the degree 2 linear strand is a scroll and the fibers of the scroll cut out a pencil of divisors on X . This connection between pencils and divisors was one of the motivations for Green's conjecture. The syzygy variety of a rank $p + 2$ -syzygy in the degree 2 linear strand is the union of $L(s)$ with a linear section of the Grassmannian $G(2, p + 3)$. The universal subbundle on the Grassmannian restricts to a special rank 2 bundle on X with many sections. For $K3$ -Surfaces this is the

Date: February 1, 2008.

Supported by Marie Curie Fellowship HPMT-CT-2001-001238.

Lazarsfeld-Mukai-bundle which has been used so successfully by Voisin in her proof of Green's conjecture.

In this note we consider determinantal varieties which are cut out by the maximal minors of a matrix of linear forms M . Eisenbud and Popescu have studied syzygies of these varieties in [EP99]. These syzygies can have arbitrary rank. Eisenbud and Popescu consider the syzygy ideal $I(s)$ of a syzygy s , which can be intrinsically defined and whose vanishing locus is the syzygy variety $\text{Syz}(s)$. They show that M is 1-generic, if and only if all last syzygies s of *minimal rank* satisfy $I(s) = I_X$. They conjecture this even holds for *all* last syzygies. For 1-generic $2 \times m$ -Matrices they show that this is true.

Here we show a similar geometric statement for arbitrary matrices M , namely that M is 1-generic if and only if all last syzygy varieties have the same support as X . This is done by evaluating the twisted $p - 1$ -forms associated to last syzygies at points outside of X using the theory of exterior minors of Green [Gre99]. By calculating the tangent spaces of X and $\text{Syz}(s)$ in a smooth point of X we can also show, that all last syzygy varieties have the same smooth locus as X .

To obtain the conjecture of Eisenbud and Popescu one would have to show, that $\text{Syz}(s)$ has no embedded components in the singular locus of X and that the syzygy ideal $I(s)$ is always saturated.

The paper has two sections. In the first the definition and properties of syzygy varieties and syzygy ideals are reviewed. The second section contains the definition and properties of 1-generic matrices and the proof of our theorems.

2. SYZYGIES AND SYZYGY VARIETIES

Let $X \subset \mathbb{P}^n$ be any projective variety over \mathbb{C} . We denote its minimal free resolution by

$$\mathcal{O}_X \leftarrow F_\bullet$$

where we consider F_\bullet as a bounded chain complex

$$F_\bullet: F_0 \leftarrow F_1 \leftarrow \dots \leftarrow 0$$

with cohomology \mathcal{O}_X concentrated in degree 0.

Since F_\bullet is graded and free we can write

$$F_p = \bigoplus_q F_{pq} \otimes \mathcal{O}(-p-q)$$

with F_{pq} vector spaces. The dimensions

$$\beta_{pq} = \dim F_{pq}$$

are called graded Betti numbers of X . Sometimes we will write more shortly

$$F_p = \bigoplus_q \mathcal{O}(-p-q)^{\beta_{pq}}$$

or collect the graded Betti numbers β_{pq} in a so-called Betti diagram:

$$\begin{array}{c|cccc} & \beta_{00} & \beta_{10} & \beta_{20} & \dots \\ & \beta_{01} & \beta_{11} & \beta_{21} & \\ & \vdots & & & \\ & & & & \beta_{pq} \end{array}$$

For better readability we will write a dash (“-”) if $\beta_{pq} = 0$.

Example 2.1. The rational normal curve $X \subset \mathbb{P}^3$ of degree 3 has minimal free resolution

$$0 \leftarrow \mathcal{O}_X \leftarrow \mathcal{O}_{\mathbb{P}^3} \leftarrow \mathcal{O}_{\mathbb{P}^3}(-2)^3 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^2 \leftarrow 0.$$

The corresponding Betti diagram is therefore

$$\begin{array}{ccc} 1 & - & - \\ - & 3 & 2 \end{array}$$

Notice that this notation corresponds to the convention used by the computer program Macaulay [GS].

Remark 2.2. By the minimality of the resolution we have

$$F_{pq} = \text{Tor}_p(S_X, \mathbb{C})_{p+q}$$

where S_X is the coordinate ring of X .

Remark 2.3. The syzygy spaces F_{pq} can also be calculated via Koszul cohomology as in [Gre84]:

$$F_{pq} = K_{pq}(S_X, V) = K_{p-1, q+1}(I_X, V)$$

where $V = H^0(\mathbb{P}^n, \mathcal{O}(1))$ is the space of linear forms on \mathbb{P}^n and $K_{pq}(B, V)$ is the cohomology of the Koszulcomplex

$$\dots \rightarrow \Lambda^{p+1}V \otimes B_{q-1} \rightarrow \Lambda^pV \otimes B_q \rightarrow \Lambda^{p-1}V \otimes B_{q+1}$$

for any $S(V)$ -module B .

The last remark gives a geometric interpretation for certain syzygies:

Lemma 2.4. *If $X \subset \mathbb{P}^n$ is not contained in any hypersurface of degree d , i.e. $(I_X)_d = 0$, then*

$$F_{p,d} = H^0(\Omega_{\mathbb{P}^n}^{p-1} \otimes I_X(d+p)).$$

In particular a p -th syzygy $s \in F_{p,d}$ can be interpreted as a twisted $(p-1)$ -form that vanishes on X .

Proof. By Koszulchomology we have

$$\begin{aligned} F_{p,d} &= K_{p-1, d+1}(I_X, V) \\ &= \ker \left(\Lambda^{p-1}V \otimes H^0(I_X(d+1)) \rightarrow \Lambda^{p-2}V \otimes H^0(I_X(d+2)) \right) \end{aligned}$$

since $H^0(I_X(d)) = 0$. This kernel can easily be identified with $H^0(\Omega_{\mathbb{P}^n}^{p-1} \otimes I_X(d+p))$ by considering exterior powers of the Euler sequence. \square

Remark 2.5. The syzygies in the above lemma are the syzygies of the first non zero row in the Betti diagram

Often the twisted $p - 1$ form associated to a syzygy does vanish on a larger variety, that contains X . This leads to the following definition, introduced by Ehbauer [Ehb94]:

Definition 2.6. Let $s \in F_{pq}$ be a p -th syzygy of $X \subset \mathbb{P}^n$ with $(I_X)_d = 0$. Then the syzygy scheme $\text{Syz}(s)$ of s is the vanishing locus of the corresponding twisted $(p - 1)$ -form.

The ideal of a syzygy scheme can be calculated via the exterior algebra structure of the linear strands in the minimal free resolution of X :

Definition 2.7. Let $s \in F_{pd}$ be a p -th syzygy of $X \subset \mathbb{P}^n$ with $(I_X)_d = 0$. Then the ideal

$$I(s) := s \wedge \Lambda^{p-1} V^* \subset (I_X)_{d+1}$$

is called the *syzygy ideal* of s .

Proposition 2.8. $V(I(s)) = \text{Syz}(s)$

Proof. Choose a basis v_i of V and let v_α be the corresponding basis of $\Lambda^{p-1} V$. Via the inclusion

$$F_{pd} \hookrightarrow \Lambda^{p-1} V \otimes (I_X)_{d+1}$$

we obtain a unique representation of s as sum

$$s = \sum_{\alpha} f_{\alpha} \otimes v_{\alpha}.$$

The f_{α} are sometimes called the polynomials involved in s with respect to the chosen basis v_{α} . They generate the syzygy ideal of s by definition. On the other hand s vanishes as a twisted $(p - 1)$ -form if and only if all f_{α} vanish. \square

Remark 2.9. The syzygy ideal is not necessarily reduced or even saturated.

Closely related to this discussion is the notion of linear forms involved in a syzygy:

Definition 2.10. Let $s \in F_{pd}$ be a p -th syzygy of $X \subset \mathbb{P}^n$ with $(I_X)_d = 0$. Then

$$s \wedge \Lambda^{p-2} V^* \otimes (I_X)_{d+1}^* \subset V$$

is called the *space of linear forms involved in s* . The linear space cut out by these linear forms is called $L(s)$. The codimension of $L(s)$ in $\mathbb{P}(V)$ is called the *rank* of s .

Remark 2.11. We only mention this definition for reference. Since the methods of this paper work for syzygies of arbitrary rank, we will not need it here.

3. TRIPLE TENSORS AND 1-GENERIC MATRICES OF LINEAR FORMS

Let A , B and C be finite dimensional vector spaces of dimensions a , b and c together with a linear map

$$\gamma: A \otimes B \rightarrow C.$$

γ can be interpreted as a triple tensor $\gamma \in A^* \otimes B^* \otimes C$ or after choosing bases as an $a \times b$ -matrix of linear forms on $\mathbb{P}(C)$. Here we adhere to the Grothendieck convention of interpreting elements of $\mathbb{P}(C)$ as linear forms on C or equivalently the elements of C as linear forms on $\mathbb{P}(C)$.

Definition 3.1. A non zero linear map $\mathbb{C} \rightarrow A$ is called a *generalized row index* of γ since it induces a map

$$\mathbb{C} \otimes B \rightarrow C$$

which can be interpreted, up to a constant factor, as a $1 \times b$ row vector of linear-forms.

If $\mathbb{C} \rightarrow A$ is such a generalized row index, the image of \mathbb{C} in A under this map is a line. We will call these images *generalized rows*. The generalized rows form a projective space $\mathbb{P}(A^*)$ which we call the *row space* of γ . Similarly $\mathbb{P}(B^*)$ is the *column space* of γ .

On the row space $\mathbb{P}(A^*)$ the triple tensor γ induces a map of vector bundles

$$\gamma_A: \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \rightarrow C$$

by composing γ with the first map of the twisted Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \rightarrow A \otimes B \rightarrow \mathbb{T}_{\mathbb{P}(A^*)}(-1) \otimes B \rightarrow 0$$

on $\mathbb{P}(A^*)$. Similarly we have

$$\gamma_B: A \otimes \mathcal{O}_{\mathbb{P}(B^*)}(-1) \rightarrow C$$

on the column space $\mathbb{P}(B^*)$. From now on we will restrict our discussion to the row space $\mathbb{P}(A^*)$, leaving the analogous constructions for the column space $\mathbb{P}(B^*)$ to the reader.

Given a generalized row $\alpha \in \mathbb{P}(A^*)$ the restriction of γ_A to α

$$\gamma_\alpha: B \rightarrow C$$

is a map of vector spaces.

Definition 3.2. The *rank* of a generalized row α is defined as $\text{rank } \alpha := \text{rank } \gamma_\alpha$. The image $\text{Im}(\gamma_\alpha) \subset C$ is called the *space of linear forms on $\mathbb{P}(C)$ involved in α* .

Remark 3.3. The determinantal varieties associated to γ_A stratify the row space $\mathbb{P}(A^*)$ according to the rank of the rows. In particular the minimal rank rows form a closed subscheme $Y_{\min} \subset \mathbb{P}(A^*)$.

Remark 3.4. In practice Y_{\min} is often not of expected codimension.

$$\gamma: A \otimes B \rightarrow C$$
$$\begin{array}{lll} \gamma(a_1 \otimes b_1) = c_1 & \gamma(a_1 \otimes b_2) = c_2 & \gamma(a_1 \otimes b_3) = c_3 \\ \gamma(a_2 \otimes b_1) = c_2 & \gamma(a_2 \otimes b_2) = c_3 & \gamma(a_2 \otimes b_3) = c_4 \end{array}$$
$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix}.$$
$$\gamma_A: \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \rightarrow C$$
$$\begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & a_1 & a_2 & 0 \\ 0 & 0 & a_1 & a_2 \end{pmatrix}.$$

Definition 3.6 (1-generic spaces). A triple tensor

$$\gamma: A \otimes B \rightarrow C$$

Example 3.7. The 2×3 -matrix considered above is 1-generic.

$$\gamma_C: A \otimes \mathcal{O}_{\mathbb{P}(C)}(-1) \rightarrow B^*$$
$$I_X \leftarrow E.$$
$$E_i = E_{ib} \otimes \mathcal{O}(-i-b), \quad E_{ib} = \Lambda^{b+i} A \otimes \Lambda^b B \otimes S_i B.$$

Green has observed, that the exterior minors of a 1-generic matrix also behave nicely:

$$\Lambda^n A \otimes S_n B \hookrightarrow \Lambda^n(A \otimes B) \xrightarrow{\quad} \Lambda^n C$$

$\underbrace{\hspace{10em}}_{e_n}$

obtained by taking the n th exterior power of γ . Then the elements in the image of e_n are called degree n exterior minors of γ .

Proposition 3.9 (Green). *If γ is 1-generic, then e_a is injective.*

Proof. [Gre99, Proposition 1.2] □

Lets now consider the last syzygies of X_{b-1} :

Definition 3.10. The syzygies $s \in E_{a-b,b}$ are called last syzygies of X_{b-1} since $E_{a-b,b}$ is the last nonzero syzygy space of X_{b-1} .

The representation of a last syzygy of X_{b-1} as an element in the corresponding Koszul cohomology group can be given explicitly:

Lemma 3.11 (Eisenbud, Popescu). *The inclusion*

$$E_{a-b,b} \hookrightarrow \Lambda^{a-b}C \otimes (I_X)_b$$

is given by the composition

$$\begin{array}{c} E_{a-b,b} \xlongequal{\quad} \Lambda^a A \otimes \Lambda^b B \otimes S_{a-b} B \\ \downarrow \\ \Lambda^b A \otimes \Lambda^b B \otimes \Lambda^{a-b} A \otimes S_{a-b} B \\ \downarrow \text{id} \otimes e_{a-b} \\ \Lambda^b A \otimes \Lambda^b B \otimes \Lambda^{a-b} C \xlongequal{\quad} (I_X)_b \otimes \Lambda^{a-b} C. \end{array}$$

Proof. [EP99, Theorem 2.1 and proof of Theorem 3.1] □

Using this Eisenbud and Popescu prove

Theorem 3.12 (Eisenbud, Popescu). *Let $X_{b-1}(\gamma_C) \subset \mathbb{P}(C)$ be a determinantal variety of expected dimension defined by a linear map*

$$\gamma: A \otimes B \rightarrow C.$$

If for all last syzygies $s \in E_{a-b,b}$ we have $I(s) = I_X$ then γ is 1-generic.

They conjecture the converse

Conjecture 3.13 (Eisenbud, Popescu). *Let $\gamma: A \otimes B \rightarrow C$ be a 1-generic map of vector spaces. Then all last syzygies s of the corresponding determinantal variety $X := X_{b-1} \subset \mathbb{P}^n$ have the same syzygy ideal $I(s) = I_X$.*

which they can prove for $b = 2$. Evaluating last syzygies directly, we obtain a geometric version of this conjecture:

Theorem 3.14. *Let $\gamma: A \otimes B \rightarrow C$ be a 1-generic map of vector spaces. Then all syzygy varieties $\text{Syz}(s)$ of last syzygies s of the corresponding determinantal variety $X := X_{b-1} \subset \mathbb{P}^n$ have the same support $\text{supp Syz}(s) = \text{supp } X$.*

Proof. Let $x \in \mathbb{P}^n$ a point not contained in X_{b-1} and s any last syzygy. We have to prove, that s does not vanish in x .

Since x is not in X_{b-1} the map γ_C has full rank in x . Therefore we can choose bases of A , B and C such that γ_C can be represented by a matrix of linear forms

$$M = \begin{pmatrix} c_{11} & \cdots & c_{1b} \\ \vdots & & \vdots \\ c_{a1} & \cdots & c_{ab} \end{pmatrix}$$

such that

$$M(x) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Now by the description of Eisenbud and Popescu above, the representation of a last syzygy s in this basis is

$$s = \sum_{|\beta|=b} f_\beta \otimes g_{\bar{\beta}}$$

where f_β is the $b \times b$ -minor involving the rows β_1, \dots, β_b of M and $g_{\bar{\beta}}$ is a degree $a - b$ exterior minor of the remaining $(a - b) \times b$ matrix. At x all minors of M except $f_{1,2,\dots,b}(x) = 1$ vanish, and

$$s(x) = g_{b+1,\dots,a}.$$

Since $g_{b+1,\dots,a}$ is a degree $a - b$ exterior minor of a 1-generic $(a - b) \times b$ Matrix it is nonzero by Green's proposition 3.9 above. \square

Interestingly the converse of this theorem is also true, strengthening the theorem of Eisenbud and Popescu.

Theorem 3.15. *Let $\gamma: A \otimes B \rightarrow C$ be a map of vector spaces, such that the corresponding determinantal variety $X := X_{b-1}$ has expected codimension and for every last syzygy s of X we have $\text{supp Syz}(s) = \text{supp } X$. Then γ is 1-generic.*

Proof. Suppose γ is not 1-generic. Then there exists a generalized row α of rank at most $b - 1$. Let $A' \subset A$ be a $a - 1$ dimensional subspace that does not contain α , and consider the induced map

$$\gamma': A' \otimes B \rightarrow C.$$

If the associated γ'_C was of submaximal rank everywhere, γ' would have a generalized column of zeros. The corresponding generalized column of γ_C would then be of maximal rank 1, i.e. the vanishing set of this column is at most of codimension 1. Since $\text{supp } X_{b-1}$ contains the vanishing set of each generalised column, this would imply that X_{b-1} has a component of codimension at most 1, which contradicts our assumption that X_{b-1} has

expected codimension. Consequently there is a point $x \in \mathbb{P}^n$ outside of X_{b-1} such that each matrix M' representing γ'_C has full rank in x .

We can therefore choose bases of A , B and C such that

- (1) M has the form

$$M = \begin{pmatrix} 0 & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & & \vdots \\ * & * & \dots & * \end{pmatrix}$$

- (2) M' consists of the last $a - 1$ rows of M

- (3) $M(x)$ has the form

$$M(x) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

Now consider the syzygy $s = (b_1)^{a-b}$ where b_1 is the basiselement of B corresponding to the first column. When we evaluate s at x we obtain $s(x) = f_{a-b+1\dots a}(x) \otimes g_{1\dots a-b,s} = g_{1\dots a-b,s}$ since $f_{a-b+1\dots a}(x) = 1$ is the only nonzero maximal minor of $M(x)$.

The exterior minor of the upper $(a - b) \times b$ submatrix corresponding to $s = (b_1)^{a-b}$ is the wedge product of the first $a - b$ linearforms in the first column of M . This wedge product vanishes since the first of these linear forms is identically zero by property (1). So s is a syzygy whose syzygy variety has support outside of X_{b-1} . \square

Our methods also allow us to describe the smooth locus of all syzygy varieties:

Theorem 3.16. *Let $\gamma: A \otimes B \rightarrow C$ be a 1-generic map of vector spaces. Then all syzygy varieties $\text{Syz}(s)$ of last syzygies s of the corresponding determinantal variety $X := X_{b-1} \subset \mathbb{P}^n$ have the same smooth locus, which is also the smooth locus of X .*

Proof. Let s be any last syzygy of X . Since $I(s) \subset I_X$ by definition and $\text{supp } X = \text{supp } \text{Syz}(s)$ by theorem 3.14, we know that the smooth locus of $\text{Syz}(s) = V(I(s))$ is contained in the smooth locus of X .

For the converse Let $x \in \mathbb{P}^n$ be a point contained in the smooth locus of X . We have to prove, that the tangent space of $\text{Syz}(s)$ in x is the same as the tangent space of X in x .

Since x is in the smooth locus of X_{b-1} the morphism γ_C has rank $b - 1$ in x . Therefore we can choose bases of A , B and C such that γ_C can be

represented by a matrix of linear forms

$$M = \begin{pmatrix} c_{11} & \dots & c_{1b} \\ \vdots & & \vdots \\ c_{a1} & \dots & c_{ab} \end{pmatrix}$$

such that

$$M(x) = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Now suppose $x + \epsilon y$ is a tangent vector of X at x . Then all maximal minors of $M(x + \epsilon y)$ have to vanish, in particular those that contain the first $b - 1$ rows and the i -th row ($i \geq b$):

$$0 = \det \begin{pmatrix} 1 + \epsilon c_{11}(y) & \dots & \epsilon c_{1,b-1}(y) & \epsilon c_{1b}(y) \\ \vdots & \ddots & \vdots & \vdots \\ \epsilon c_{b-1,1}(y) & \dots & 1 + \epsilon c_{b-1,b-1}(y) & \epsilon c_{b-1,b}(y) \\ \epsilon c_{i,1}(y) & \dots & \epsilon c_{i,b-1}(y) & \epsilon c_{ib}(y) \end{pmatrix} = \epsilon c_{ib}(y).$$

All other minors vanish since every term of the corresponding determinant involves at least ϵ^2 . So $x + \epsilon y$ is tangent to X if and only if $c_{bb}(y) = \dots = c_{ab}(y) = 0$.

Now assume that $x + \epsilon y$ is not a tangent vector of X . Then we can assume after a another base change of C , that $M(x + \epsilon y)$ has the form

$$M(x + \epsilon y) = \begin{pmatrix} 1 + \epsilon c_{11}(y) & \dots & \epsilon c_{1,b-1}(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \epsilon c_{b-1,1}(y) & \dots & 1 + \epsilon c_{b-1,b-1}(y) & 0 \\ \epsilon c_{b,1}(y) & \dots & \epsilon c_{b,b-1}(y) & \epsilon \\ \epsilon c_{b+1,1}(y) & \dots & \epsilon c_{b+1,b-1}(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \epsilon c_{a1}(y) & \dots & \epsilon c_{a,b-1}(y) & 0 \end{pmatrix}$$

As before the representation of a last syzygy s in this basis is

$$s = \sum_{|\beta|=b} f_\beta \otimes g_{\bar{\beta}}$$

where f_β is the $b \times b$ -minor involving the rows β_1, \dots, β_b of M and $g_{\bar{\beta}}$ is a degree $a - b$ exterior minor of the remaining $a - b \times b$ matrix. At $x + \epsilon y$ all minors of M except $f_{1,2,\dots,b}(x) = \epsilon$ vanish, and

$$s(x) = \epsilon g_{b+1,\dots,a}.$$

Since $g_{b+1,\dots,a}$ is again a degree $a - b$ exterior minor of a 1-generic $(a - b) \times b$ matrix it is nonzero by Green's proposition 3.9 above. Therefore $x + \epsilon y$ is not a tangent vector of $\text{Syz}(s)$. This shows that the tangent space of $\text{Syz}(s)$ at x is contained in the tangent space of X at x . Since on the other hand $\text{Syz}(s)$ contains X as scheme both tangent spaces have to coincide. \square

REFERENCES

- [Buc70] B. Buchberger. Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. *Aequationes Math.*, 4:374–383, 1970.
- [Ehb94] S. Ehbauer. Syzygies of points in projective space and applications. In F. Orecchia, editor, *Zero-dimensional schemes. Proceedings of the international conference held in Ravello, Italy, June 8-13, 1992*, pages 145–170, Berlin, 1994. de Gruyter.
- [Eis88] D. Eisenbud. Linear sections of determinantal varieties. *Amer. J. Math.*, 110(3):541–575, 1988.
- [EP99] David Eisenbud and Sorin Popescu. Syzygy ideals for determinantal ideals and the syzygetic Castelnuovo lemma. In *Commutative algebra, algebraic geometry, and computational methods (Hanoi, 1996)*, pages 247–258. Springer, Singapore, 1999.
- [ES94] F. Eusei and F.O. Schreyer. A remark to a conjecture of Paranjape and Ramanan. <http://btm8x5.mat.uni-bayreuth.de/~schreyer/>, 1994.
- [Gre84] M.L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differential Geometry*, 19:125–171, 1984.
- [Gre99] M.L. Green. The Eisenbud-Koh-Stillman conjecture. *Inv. Math.*, 136:411–418, 1999.
- [GS] Daniel R. Grayson and Michael E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2>.
- [Sch80] F.O. Schreyer. Die Berechnung von Syzygien mit dem verallgemeinerten Weierstrass’schen Divisionssatz. Diploma thesis, University of Hamburg, Germany, 1980.
- [Sch91] F.O. Schreyer. A standard basis approach to syzygies of canonical curves. *J. reine angew. Math.*, 421:83–123, 1991.
- [vB01] H.-Chr. Graf v. Bothmer. Geometric syzygies of canonical curves of even genus lying on a $K3$ -surface. *math.AG/0108078*, 2001.
- [Voi02] Claire Voisin. Green’s generic syzygy conjecture for curves of even genus lying on a $K3$ surface. *J. Eur. Math. Soc. (JEMS)*, 4(4):363–404, 2002.
- [Voi03] C. Voisin. Green’s canonical syzygy conjecture for generic curves of odd genus. *math.AG/0301359*, 2003.

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